Ryan Chilton
ECE 311 HW #2 (Solutions)

(2.5) Use the R' L' G' C' parameters from HW #1 solutions (available online on course webpage)

$R' = 0.788\ \Omega/m$

$L' = 139\ nH/m$ $\qquad \omega = 1\,GHz \times 2\pi$

$G' = 9.1\,mS/m$

$C' = 36.1\ pF/m$

$$\gamma = \sqrt{(R'+j\omega L')(G'+j\omega C')} = 0.28 + j14.08$$

$$\alpha = Re\{\gamma\} = 0.287 \text{ nepers/m}$$

$$\beta = Im\{\gamma\} = 14.08 \text{ rad/m}$$

$$u_p = \frac{\omega}{\beta} = 4.46 \times 10^8 \text{ m/s}$$

This exceeds the speed of light! This means my calculations for L', R', G', C' are suspect. I think it's L' that is wrong. My $L' < \mu_0$, I don't think is possible. It's ok if you use my wrong numbers as inputs. Anyway...

$$Z_0 = \sqrt{\frac{R'+j\omega L'}{G'+j\omega C'}} = (62 + 1.2j)\,\Omega$$

(2.6) Compute the parameters α, β and Z_0 for a distortionless line, which satisfies $R'C' = L'G'$.

a) Compute α, β

Note that if $R'C' = L'G'$, then $\frac{R'}{L'} = \frac{G'}{C'}$. Denote this ratio as "K". Use the following formula for γ^2:

$$\gamma^2 = (R' + j\omega L')(G' + j\omega C')$$

$$\gamma^2 = L'\left(\frac{R'}{L'} + j\omega\right) C'\left(\frac{G'}{C'} + j\omega\right) = L'C'(K + j\omega)(K + j\omega)$$

Easy to take the square root of this:

$$\gamma = (K + j\omega)\sqrt{L'C'} = K\sqrt{L'C'} + j\omega\sqrt{L'C'}$$

Now, seperate the real and imaginary parts. By def., $\alpha = Re\{\gamma\}$, $\beta = Im\{\gamma\}$

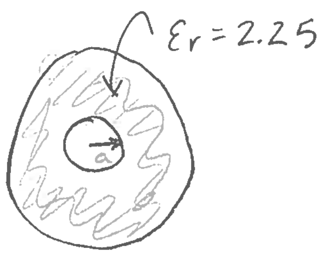
$$\alpha = K\sqrt{L'C'} = \sqrt{K^2 L'C'} = \sqrt{K \cdot L' \cdot K \cdot C'}$$

$$= \sqrt{\underbrace{\frac{R'}{L'}}_{K = R'/L'} \cdot L' \cdot \underbrace{\frac{G'}{C'}}_{K = G'/C'} \cdot C'} = \sqrt{R' \cdot G'}$$

b) Compute Z_0 - ~~uses~~ the same trick

$$Z_0 = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}} = \sqrt{\frac{\left(\frac{R'}{L'} + j\omega\right) L'}{\left(\frac{G'}{C'} + j\omega\right) C'}} = \sqrt{\frac{\cancel{(k + j\omega)} L'}{\cancel{(k + j\omega)} C'}} = \sqrt{\frac{L'}{C'}}$$

(2.11) Lossless coaxial line, $Z_0 = 50\Omega$, a = inner radius = 1.2 mm



(a) Find the radius of the outer conductor

From Table 2-2, page 55:

$$Z_0 = \left(\frac{60}{\sqrt{\varepsilon_r}}\right) \ln(b/a) \quad \leftarrow \text{solve for } b.$$

$$\exp\left(\frac{Z_0 \cdot \sqrt{\varepsilon_r}}{60\Omega}\right) \cdot a = b \quad \leftarrow \text{substitute known parameters}$$

$$4.19 \text{ mm} = b$$

(b) Find the phase velocity of the line:

$$u_p = \frac{c}{\sqrt{\varepsilon_r}} = \frac{3\times10^8 \text{ m/s}}{\sqrt{2.25}} = 2\times10^8 \text{ m/s} \quad (2/3 \times c)$$

(2.9) Transmission line - general case - find R', L', C', G'!

$\omega = 125 \text{ MHz} \times 2\pi$

$Z_0 = 40\Omega$

$\alpha = 0.02$ Np/m

$\beta = 0.75$ rad/m

} these give us γ (complex propag. const)

What do we know?

$$Z_0 = \sqrt{\frac{R'+j\omega L'}{G'+j\omega C'}} \quad \text{OR} \quad Z_0^2 = \left(\frac{R'+j\omega L'}{G'+j\omega C'}\right)$$

$$\gamma = \sqrt{(R'+j\omega L')(G'+j\omega C')} \quad \text{OR} \quad \gamma^2 = (R'+j\omega L')(G'+j\omega C')$$

Define $K_1 := (R'+j\omega L')$ and $K_2 := (G'+j\omega C')$. Then we have:

$$Z_0^2 = \frac{K_1}{K_2}$$

$$\gamma^2 = K_1 \cdot K_2$$

} Next solve this 2 eq. 2 unk. system for K_1 & K_2 $\Rightarrow$

$$Z_0^2 \cdot \gamma^2 = \left(\frac{K_1}{K_2}\right) K_1 K_2 = K_1^2$$

$$\gamma^2 / Z_0^2 = \frac{K_1 K_2}{(K_1/K_2)} = K_2^2$$

So now we have:

$$Z_0^2 \cdot \gamma^2 = K_1^2 \quad \text{OR} \quad Z_0 \cdot \gamma = K_1$$

$$\gamma^2 / Z_0^2 = K_2^2 \quad \text{OR} \quad \gamma / Z_0 = K_2$$

Now we can compute K_1 and K_2.

(Con't)

$$K_1 = Z_0 \cdot \gamma = 40\Omega \times (0.02 + 0.75j) = 0.8 + j30$$

$$K_2 = \gamma / Z_0 = \frac{(0.02 + j0.75)}{40\Omega} = 5 \times 10^{-3} + 1.875 \times 10^{-2}$$

Use these values, and the operating frequency, to deduce R'L'C'

$$K_1 = R' + j\omega L' = 0.8 + j30$$

match real parts

match imag. parts

$\therefore$ $R' = 0.8\,\Omega/m$

$\omega L' = 30$ $\left(\omega = \frac{125\,\text{MHz}}{2\pi}\right)$

$L' = \frac{30}{\omega} = 38.2\,nH/m$

$$K_2 = G' + j\omega C' = 5 \times 10^{-3} + j1.875 \times 10^{-2}$$

$\therefore$ $G' = 5\,mS/m$

$\omega C' = 1.875 \times 10^{-2}$

$C' = 23.9\,pF/m$

2.15) Given a load $Z_L = (25 - j50)\Omega$, find a characteristic impedance Z_0 that minimizes the standing wave ratio. The line must be lossless (Z_0 purely-real).

VSWR and $\tilde{\Gamma}$ are both functions of Z_0

$$VSWR(Z_0) = \frac{1 + |\tilde{\Gamma}(Z_0)|}{1 - |\tilde{\Gamma}(Z_0)|} = \frac{1 + \sqrt{\tilde{\Gamma}(Z_0)\tilde{\Gamma}^*(Z_0)}}{1 - \sqrt{\tilde{\Gamma}(Z_0)\tilde{\Gamma}^*(Z_0)}}$$ ← minimize this function

where

$$\tilde{\Gamma}(Z_0) = \frac{\tilde{Z}_L - Z_0}{\tilde{Z}_L + Z_0} = \frac{(25 - Z_0) - j50}{(25 + Z_0) - j50}$$

Similarly, the conjugate is just:

$$\tilde{\Gamma}^*(Z_0) = \frac{(25 - Z_0) + j50}{(25 + Z_0) + j50}$$

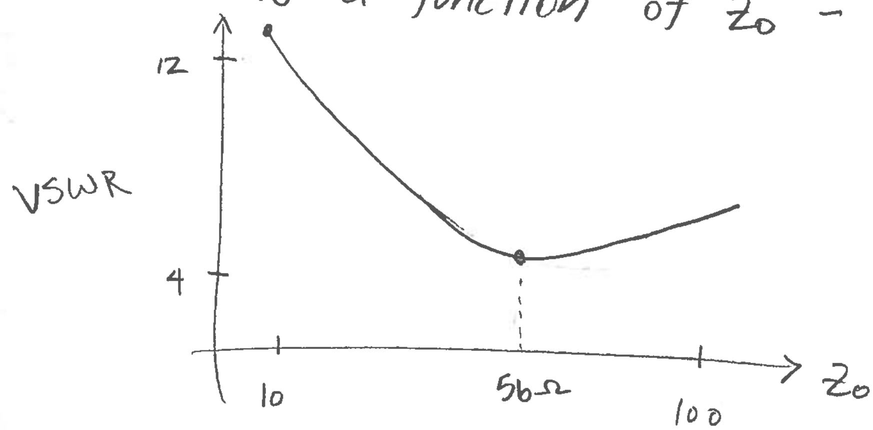
$$\tilde{\Gamma}\tilde{\Gamma}^* = \frac{[(25 - Z_0) - j50][(25 - Z_0) + j50]}{[(25 + Z_0) - j50][(25 + Z_0) + j50]} = \frac{(25 - Z_0)^2 + 2500}{(25 + Z_0)^2 + 2500}$$

CONT'D ↓

Substitute this into the VSWR equation

$$VSWR(Z_0) = \frac{1+\sqrt{\tilde{\Gamma}\tilde{\Gamma}^*}}{1-\sqrt{\tilde{\Gamma}\tilde{\Gamma}^*}} = \frac{1+\sqrt{\dfrac{(25-Z_0)^2+2500}{(25+Z_0)^2+2500}}}{1-\sqrt{\dfrac{(25-Z_0)^2+2500}{(25+Z_0)^2+2500}}}$$

Plot VSWR as a function of Z_0 - graphical calc. or matlab



Graphical solution (which is sufficient for me) yields $Z_0 \cong 56$
VSWR $\cong 4.24$

If you really want an exact answer, recognize that the function $VSWR = \frac{1+\sqrt{\tilde{\Gamma}\tilde{\Gamma}^*}}{1-\sqrt{\tilde{\Gamma}\tilde{\Gamma}^*}}$ is minimized when $\sqrt{\tilde{\Gamma}\tilde{\Gamma}^*}$ is minimized. This is intuitive - small $|\tilde{\Gamma}|$ yields small VSWR. Furthermore, $\sqrt{\ }$ is monotonic, so $\sqrt{\Gamma\Gamma^*}$ is min'zd when $\Gamma\Gamma^*$ is min'zd...

$$\tilde{\Gamma}\tilde{\Gamma}^* = \frac{(25-Z_0)^2+2500}{(25+Z_0)^2+2500}$$

differentiate via quotient rule

CALC I - function minimized when the derivative is zero (and the function is concave-up - but we alread know that's true from the picture)

$$\frac{\partial}{\partial Z_0}\left[\tilde{\Gamma}\tilde{\Gamma}\right] = \frac{\left[(25+Z_0)^2+2500\right]\left[-2(25-Z_0)\right] - \left[(25-Z_0)^2+2500\right]\left[2(25+Z_0)\right]}{\left[(25+Z_0)^2+2500\right]^2} =$$

Set this to zero ... just the numerator is relevant.

$$\left[(25+Z_0)^2+2500\right]\left[-2(25-Z_0)\right] - \left[(25-Z_0)^2+2500\right]\left[2(25+Z_0)\right] = 0$$

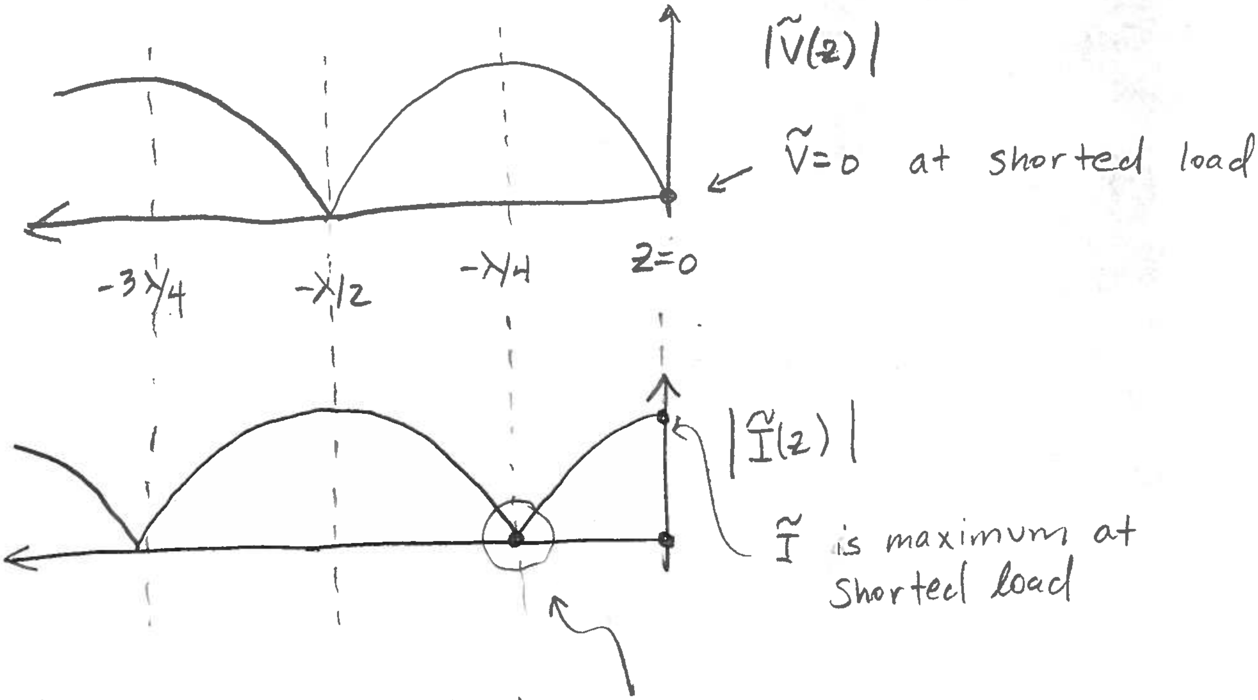
algebra is fun!

$$100\,Z_0^2 - 312{,}500 = 0 \longrightarrow Z_0 = \sqrt{3125}\ \Omega \cong 55.9\ \Omega$$

(and then VSWR = 4.236...)

2.25) A line is terminated with a short – how long should the line be to appear as an open?

We discussed this concept in class – recall the voltage and current distributions for the shorted line.



Note at $z=-\lambda/4$, the current is zero. This exactly how an open circuit behaves. Thus, the line should be $\lambda/4$.

Another way to see this is to use the impedance formula we learned on Mon 10/1 ..

$$\tilde{Z}_{in}(-\ell) = Z_0 \cdot \frac{Z_L + jZ_0\tan(\beta\ell)}{Z_0 + jZ_L\tan(\beta\ell)} \xrightarrow[\substack{\tilde{Z}_L=0 \\ \text{(short)}}]{} = Z_0 \cdot \frac{jZ_0\tan(\beta\ell)}{Z_0}$$

$$= jZ_0\tan(\beta\ell)$$

As $\beta\ell$ approaches $\pi/2$, this $\tan(\beta\ell)$ terms grows without bound, realizing infinite impedance (an open circuit). Set $\beta\ell = \pi/2$...

$$\beta\ell = \pi/2 \xrightarrow[\beta=\frac{2\pi}{\lambda}]{} \left(\frac{2\pi}{\lambda}\right)\ell = \pi/2 \xrightarrow[\text{Solve for } \ell]{} \ell = \lambda/4$$