The poisson equation and the laplace equation. (I)

Recall the relationship between electric scalar potential, V, and electric field intensity, $\vec{E}$.

$$\vec{E} = -\nabla V$$

$$= -\left[\hat{x}\frac{\partial}{\partial x}V + \hat{y}\frac{\partial}{\partial y}V + \hat{z}\frac{\partial}{\partial z}V\right]$$

Gauss' law in point form gave us a constraint that the electric flux density, $\vec{D} = \varepsilon_0\vec{E}$, must satisfy:

$\nabla \cdot \vec{D} = \rho_v$ Gauss' Law

$\nabla \cdot (\varepsilon_0 \vec{E}) = \rho_v$ Relate $\vec{D}$ & $\vec{E}$

$\nabla \cdot \left(\hat{x}\frac{\partial}{\partial x}V + \hat{y}\frac{\partial}{\partial y}V + \hat{z}\frac{\partial}{\partial z}V\right) = -\frac{\rho_v}{\varepsilon_0}$ Substitute $\vec{E} = -\nabla V$

$\left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial y}\right) \cdot \left(\hat{x}\frac{\partial}{\partial x}V + \hat{y}\frac{\partial}{\partial y}V + \hat{z}\frac{\partial}{\partial z}V\right) = \frac{-\rho_v}{\varepsilon_0}$ write $\nabla\cdot$

$$\left(\frac{\partial^2}{\partial x^2}V + \frac{\partial^2}{\partial y^2}V + \frac{\partial^2}{\partial z^2}V\right) = -\rho_v/\varepsilon_0$$

This operator is called the "scalar laplacian", ∇^2

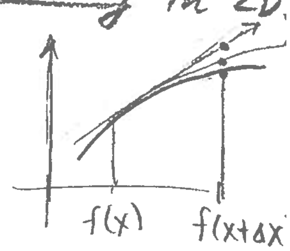
$$\nabla^2 V = -\rho_v/\varepsilon_0$$

When dealing with charge free regions, $\rho_v = 0$

$$\nabla^2 V = 0$$

This is called the "laplace equation". Today, we'll investigate how to solve this equation approximately & numerically in 2D.

$$V = V(x,y)$$



Derivatives, finite differences, & convergence

Recall the taylor expansion of a function:

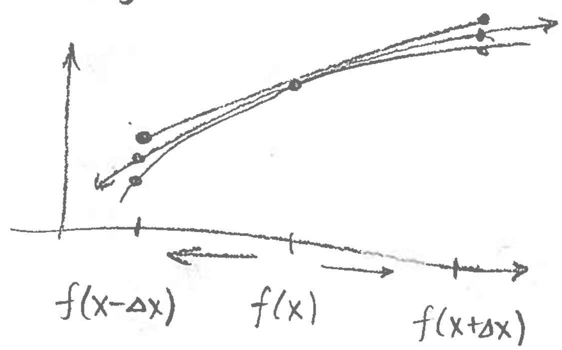
$$f(x+\Delta x) = f(x) + \Delta x \cdot f'(x) + \frac{\Delta x^2}{2!}f''(x) + \frac{\Delta x^3}{6}f'''(x) + \dots \frac{\Delta x^n}{n!}f^{(n)}(x)$$

a sampled value

You can just as easily work backwards: $+\Delta x \to -\Delta x$

$$f(x-\Delta x) = f(x) - f'(x)\Delta x + f''(x)\cdot\frac{\Delta x^2}{2} - f'''(x)\cdot\frac{\Delta x^3}{6} + \dots$$

(extrapolating backwards)



(II)

Subtract the 2ND equation from the first; term by term

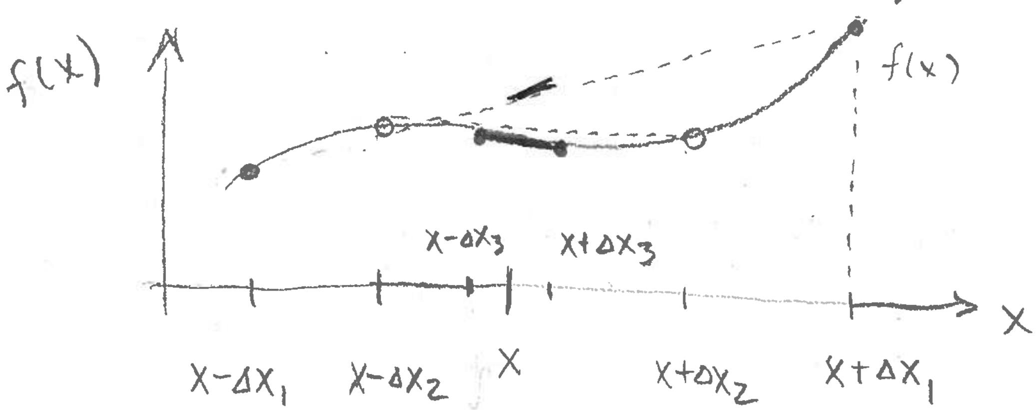
$$\begin{array}{c} f(x+\Delta x) \\ - \\ f(x-\Delta x) \end{array} = \begin{array}{c} f(x) + \Delta x\, f'(x) + \frac{\Delta x^2}{2} f''(x) + \frac{\Delta x^3}{6} f'''(x) + \dots \\ - \\ f(x) - \Delta x\, f'(x) + \frac{\Delta x^2}{2} f''(x) - \frac{\Delta x^3}{6} f'''(x) + \dots \end{array}$$

$$f(x+\Delta x) - f(x-\Delta x) = 2\Delta x\, f'(x) + \frac{2\Delta x^3}{6} f'''(x) + \dots$$

Now, solve for $f'(x)$!

$$\frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x} - \frac{\Delta x^2}{6} f'''(x) - \dots = f'(x)$$

This is an equation to estimate $f'(x)$ from sampled values of $f(x)$. The term $f'''(x)$ is an error term.



It vanishes as $\Delta x \to 0$.

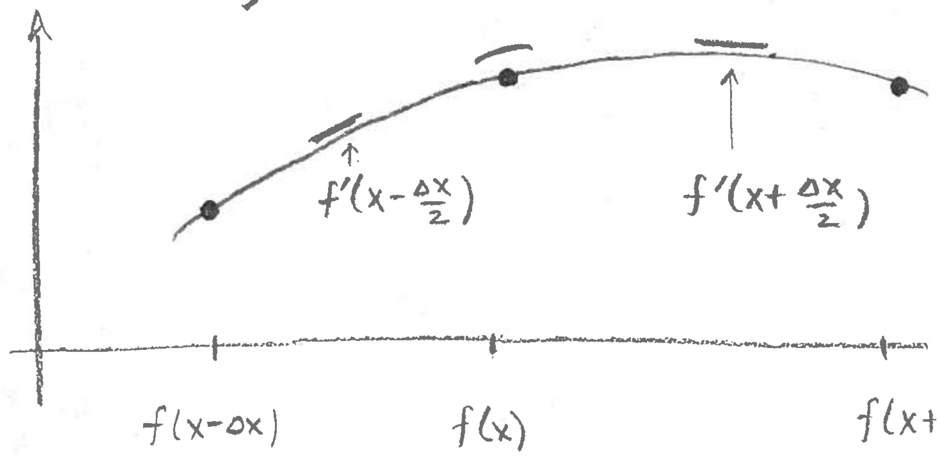
Assume Δx is really small, and drop it away.

$$\frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x} \cong f'(x)$$

Intuitive - classic interpretation of derivative as a slope.. "finite difference"

The laplacian operator contains second derivatives. What can we do to find a second derivative? 2ND derivative is the <u>derivative</u> of the <u>derivative</u> ... nest the differencing procedure:

(III)



Make two first derivative estimates:

$f''(x) = \frac{\partial}{\partial x} f'(x)$ ← 2 samples!

$$f'(x-\tfrac{\Delta x}{2}) \cong \frac{f(x) - f(x-\Delta x)}{\Delta x}$$

$$f'(x+\tfrac{\Delta x}{2}) \cong \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Now, finite difference these to estimate f'':

$$f''(x) \cong \frac{f'(x+\frac{\Delta x}{2}) - f'(x-\frac{\Delta x}{2})}{\Delta x} \cong \frac{[f(x+\Delta x) - f(x)] - [f(x) - f(x-\Delta x)]}{(\Delta x)^2}$$

$$f''(x) \cong \frac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{(\Delta x)^2}$$

You can nest this procedure as deep as you wish. In general, computing $f^{(n)}(x)$ requires $n+1$ sampled values.

Let's return to the Laplace equation in 2D:

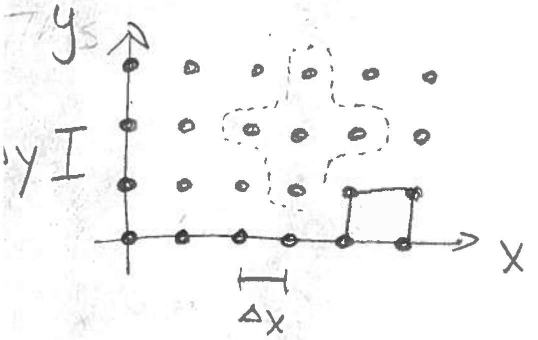
(IV)

$$V = V(x,y)$$

$$\nabla^2 V(x,y) = \frac{\partial^2}{\partial x^2} V + \frac{\partial^2}{\partial y^2} V = 0$$

Use the finite difference formula...

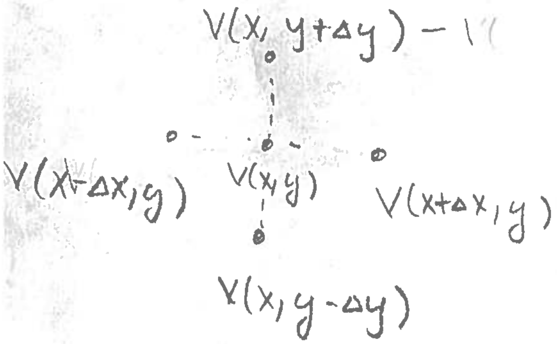
$$0 = \frac{V(x+\Delta x, y) - 2V(x,y) + V(x-\Delta x, y)}{\Delta x^2} + \frac{V(x, y+\Delta y) - 2V(x,y) + V(x, y-\Delta y)}{\Delta y^2}$$



Imagine a large, uniform grid of samples, and pick $\Delta x = \Delta y$

Multiply by $\Delta x^2 (= \Delta y^2)$

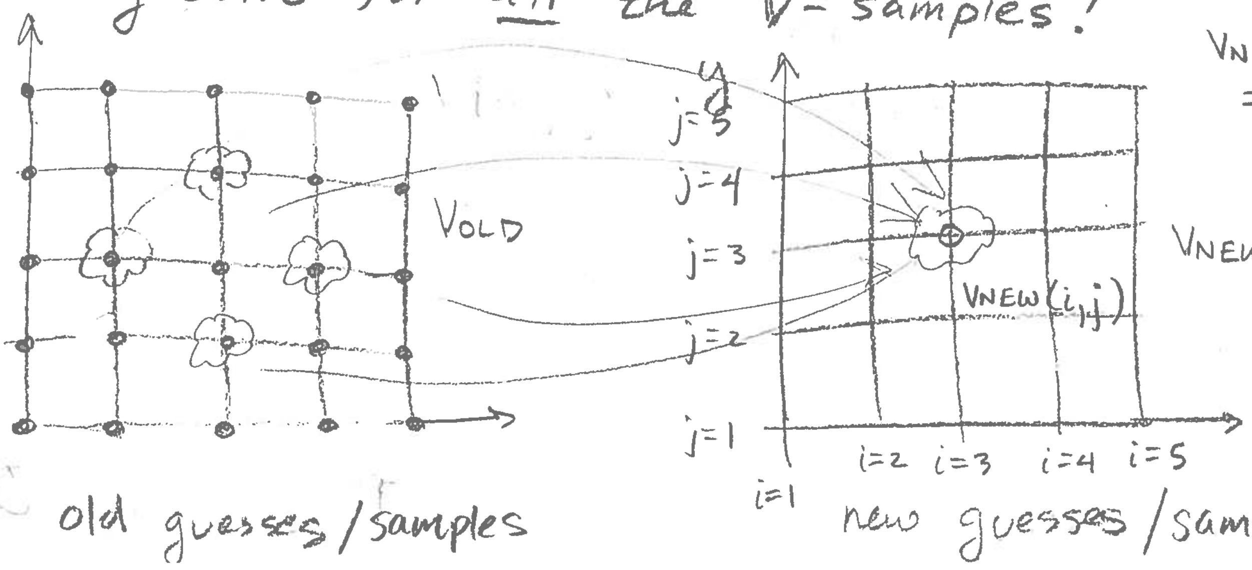
$$0 = V(x+\Delta x, y) + V(x-\Delta x, y) + V(x, y+\Delta y) + V(x, y-\Delta y) - 4V(x,y)$$



And solve for $V(x,y)$:

$$V(x,y) = \frac{1}{4}\left[V(x-\Delta x, y) + V(x+\Delta x, y) + V(x, y+\Delta y) + V(x, y-\Delta y) \right]$$

Intuitive explanation... V at a given point is the <u>average</u> of its four nearest neighbors. Use this to iteratively solve for <u>all</u> the V-samples!



$V_{NEW}(i\Delta x, j\Delta y)$

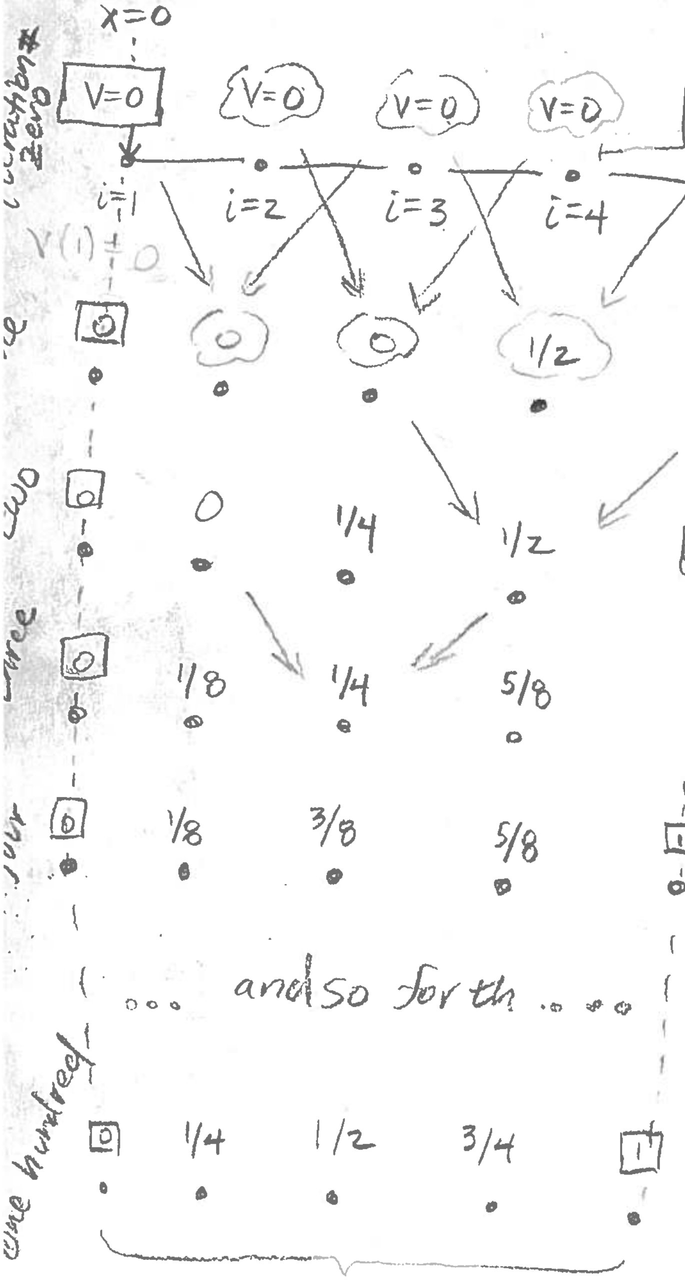
$= \frac{1}{4}[V_{OLD}(i-1, j)$

$V_{OLD}(i+1, j)$

$V_{OLD}(i, j+1)$

$V_{OLD}(i, j-1)$

Let's do a one-dimensional example; thoroughly (V)



$$V_{NEW}(i) = \frac{1}{2}\left[V_{OLD}(i-1) + V_{OLD}(i+1)\right]$$

$\boxed{\text{Boundary condition}}$

Free/unknown voltage

Sample matlab code

```
for loop=1:100
    VNEW = zeros(5,1)          Vnew(5)=1
    VOLD = zeros(5,1)          Vold(5)=1
    for i=2:4
        VNEW(i) = 1/2[Vold(i-1)+ Vold(i+1)]
    end
    VOLD= VNEW
end
```

all these voltage samples converge to the solution as you do more & more iterations!

This is a <u>discrete representation</u> of <u>true solution</u>,

$V(x) = x$ { verify for yourself that $\nabla^2 V(x) = 0$, $V(0)=0$, $V(1)=1$ }

To recap; solving the "boundary value problem"

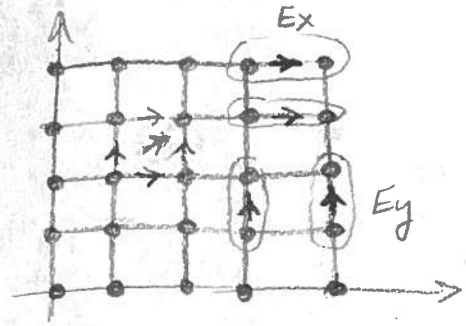
(1) $V(x,y)$ is governed by $\nabla^2 V = 0$

(3) Finite differences are a systematic way to improve your guess, over many iterations.

(2) You are given values of V which are fixed for all iterations (boundary conditions), pick some initial guess for the other V's (o

Some remarks about post processing ...

once you know V, you can compute $\vec{E} = -\nabla V$.
how do you do the differentiation? <u>Finite differences!</u>



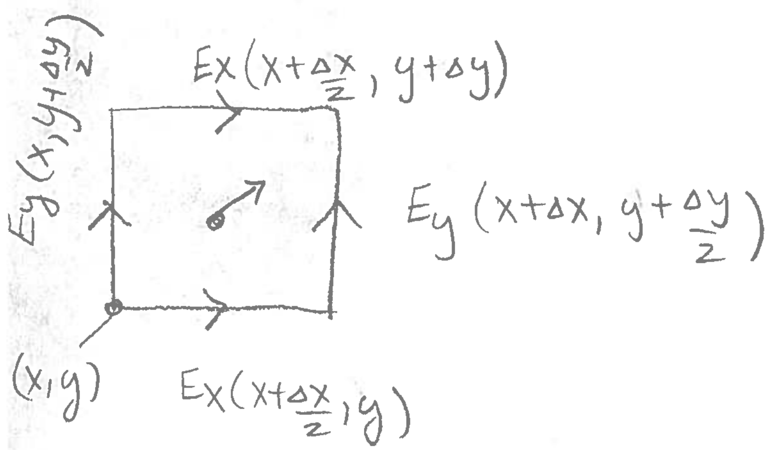
$$\vec{E} = -\nabla V$$

$$\vec{E} = -\hat{x}\frac{\partial}{\partial x}V - \hat{y}\frac{\partial}{\partial y}V$$

$$E_x\left(x+\frac{\Delta x}{2}, y\right) \cong -\frac{V(x+\Delta x, y) - V(x,y)}{\Delta x}$$

$$E_y(x, y+\Delta y/2) \cong -\frac{V(x, y+\Delta y) - V(x,y)}{\Delta y}$$

Then interpolate them at the cell centroids, to compose a field with both components



But <u>don't</u> do this!

* Use matlabs gradient command
* Use "contour" as well
* see "help gradient" for an example

You will be assigned one of the following problems: